

Histopolation Methods: Global Approximation, Local Enrichment, and Spectral Stability

Federico Nudo

Estimated duration: 16 hours.

Introduction and background requirements

This course is devoted to histopolation methods, a natural extension of classical interpolation schemes in which pointwise constraints are replaced by integral conditions. Rather than prescribing function values at discrete nodes, histopolation aims to reconstruct functions from prescribed averages or moments over subdomains, such as segments, edges, or faces. This paradigm naturally arises in applications where pointwise data are unavailable, including inverse problems, imaging, and tomography.

The first part of the course focuses on the one-dimensional setting, which provides a transparent framework for polynomial reconstruction from segments and for the rigorous analysis of unisolvence and conditioning. Within this setting, histopolation–regression methods on segments based on equispaced data are first considered. It is shown that Runge-type instability phenomena persist, as in classical interpolation theory. To overcome this limitation, three different histopolation–regression strategies based on mock-Chebyshev-type segments are introduced and analyzed. Building on the univariate theory, the analysis is then extended to multivariate settings and to general partitioned computational domains. In this context, the notions of Padua, Fekete, and Leja triangles are introduced.

The second part of the course is devoted to local approximation methods on polygonal and simplicial meshes. General enrichment strategies based on orthogonal polynomial systems are presented. This establishes a direct and systematic link between histopolation techniques and the theory of classical orthogonal polynomials. As a representative example, weighted enriched formulations based on Jacobi-type weights are introduced, enhancing approximation properties while preserving local well-posedness.

The final part of the course adopts a numerical linear algebra perspective and focuses on the spectral analysis of the structured moment matrices arising from enriched histopolation schemes. By exploiting block decompositions and Schur complement techniques, unisolvence and stability are characterized in terms of reduced operators. Their spectral properties determine the conditioning of the reconstruction process and reveal dimension-independent stability mechanisms. These concepts are further illustrated through recent results, showing that unisolvence alone does not ensure bounded conditioning and that stability crucially depends on the interplay between segment and the choice of polynomial basis.

The course is intended for PhD students in applied mathematics and related fields. A solid background in numerical analysis and linear algebra is assumed, together with familiarity with polynomial approximation methods.

Aims:

By the end of the course, students are expected to:

- understand the conceptual foundations of histopolation and its differences with classical interpolation methods;

- analyze stability, unisolvence, and approximation properties of histopolation schemes;
- construct local histopolation methods on intervals, multivariate domains, and general polygonal or simplicial meshes;
- apply weighted enrichment strategies, including Jacobi-type weighted formulations, to improve reconstruction accuracy.

Course contents:

The course will cover the following topics:

- Histopolation on intervals: polynomial reconstruction from segment data (2 hours).
- Histopolation–regression methods based on mock-Chebyshev-type segments (2 hours).
- Extension of histopolation–regression techniques to multivariate and general computational domains (2 hours).
- Local nonconforming histopolation methods on polygonal and simplicial meshes (4 hours).
- Quadratic enriched weighted histopolation methods based on Jacobi-type weights (2 hours).
- Structured moment matrices, Schur complements, and spectral characterization of stability and conditioning (4 hours).

Bibliography:

- A. Guessab, F. Nudo. Spectral Schur analysis of structured moment matrices for quadratic histopolation, arXiv:2601.00301, 2026;
- L. Bruni Bruno, S. Serra-Capizzano. On the conditioning of histopolation, arXiv:2511.15395, 2026;
- F. Dell’Accio, A. Guessab, G. V. Milovanović, F. Nudo. Nonconforming approximation methods for function reconstruction on general polygonal meshes via orthogonal polynomials, *IMA J. Numer. Anal.*, accepted;
- L. Bruni Bruno, F. Dell’Accio, W. Erb, F. Nudo. Bivariate polynomial histopolation techniques on Padua, Fekete and Leja triangles, arXiv:2506.03025, 2025;
- L. Bruni Bruno, F. Dell’Accio, W. Erb, F. Nudo. Polynomial Histopolation on Mock-Chebyshev Segments, *J. Sci. Comput.*, 104, 65, 2025;
- L. Bruni Bruno, W. Erb. Polynomial interpolation of function averages on interval segments, *SIAM J. Numer. Anal.*, 62(4), 1759–1781, 2024;
- V.P. Palamodov. Reconstruction from integral data. CRC Press Boca Raton, FL, 2016.